The Incomplete Gamma Function
Part III - A Mean-Reverting Revenue Model

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The base equation for a mean-reverting process from Part II where the variable t is time in years is... [1]

$$f(t)=\int_m^n \mathrm{Exp}\left\{d+c\,t-a\,\mathrm{Exp}\left\{-\,b\,t\right\}\right\}\delta t \quad \text{where...} \;\; a>0\;\;,\;\;b>0\;\;,\;\;c<0\;\;,\;\;n>m\geq 0 \tag{1}$$

The solution to the base equation where $\Gamma(x,y)$ is the incomplete gamma function is... [1]

$$f(t)=\mathrm{Exp}\left\{d\right\}a^{\frac{c}{b}}b^{-1}\left[\Gamma\left(-\frac{c}{b},\,a\,\mathrm{Exp}\left\{-\,b\,n\right\}\right)-\Gamma\left(-\frac{c}{b},\,a\,\mathrm{Exp}\left\{-\,b\,m\right\}\right)\right] \tag{2}$$

In this white paper we will use the base equation for a mean-reverting processes to solve the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a revenue model given the following parameters...

Table 1: Revenue Model Parameters

Description	Value
Annualized revenue at time zero (in dollars)	1,000,000
Current unsustainable revenue growth rate (%)	25.00
Long-term sustainable revenue growth rate (%)	5.00
Rate of mean reversion	0.30
After-tax revenue margin (%)	20.00
Risk-adjusted discount rate (%)	15.00

We will use our model to answer the following questions:

Question 1: Graph the revenue growth rate over the time interval $[0,20]$.

Question 2: What is the present value of net income realized in year 3?

Building The Valuation Equation

We will define the variable μ_t to be the annualized revenue growth rate at time t, the variable ω to be the long-term sustainable revenue growth rate, the variable Δ to be the difference between the current growth rate and the long-term sustainable rate, and the variable λ to be the rate of mean reversion. The equation for the annualized revenue growth rate is...

$$\mu_t=\omega+\Delta\,\mathrm{Exp}\left\{-\,\lambda\,t\right\} \tag{3}$$

We will define the variable Γ_t to be the cumulative revenue growth rate over the time interval $[0,t]$. Using Equation (3) above the equation for the cumulative revenue growth rate at time t is...

$$\Gamma_t=\int_0^t \mu_s\,\delta s=\omega\,t+\frac{\Delta}{\lambda}\left(1-\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right) \tag{4}$$

We will define the variable R_t to be annualized revenue at time t. Using Equation (4) above the equation for annualized revenue is...

$$R_t = R_0 \,\mathrm{Exp}\left\{\Gamma_t\right\} = R_0 \,\mathrm{Exp}\left\{\omega\, t + \frac{\Delta}{\lambda}\left(1 - \mathrm{Exp}\left\{-\lambda\, t\right\}\right)\right\} \tag{5}$$

We will define the variable $P_{m,n}$ to be the present value of net income realized over the time interval $[m, n]$, the variable θ to be the after-tax revenue margin, and the variable κ to be the cost of capital. Using Equation (5) above the equation for the present value of net income is...

$$\begin{aligned} P_{m,n} &= \int\limits_m^n \theta\, R_t \,\mathrm{Exp}\left\{-\kappa\, t\right\} \delta t \\ &= \theta\, R_0 \int\limits_m^n \mathrm{Exp}\left\{\omega\, t + \frac{\Delta}{\lambda}\left(1 - \mathrm{Exp}\left\{-\lambda\, t\right\}\right)\right\} \mathrm{Exp}\left\{-\kappa\, t\right\} \delta t \\ &= \theta\, R_0 \int\limits_m^n \mathrm{Exp}\left\{\frac{\Delta}{\lambda} + (\omega - \kappa)\, t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\, t\right\}\right\} \delta t \end{aligned} \tag{6}$$

We will make the following definitions...

$$a = \frac{\Delta}{\lambda} \text{ ...and... } b = \lambda \text{ ...and... } c = \omega - \kappa \text{ ...and... } d = \frac{\Delta}{\lambda} \tag{7}$$

Using the definitions in Equation (7) above we can rewrite Equation (6) above as...

$$P_{m,n} = \theta\, R_0 \int\limits_m^n \mathrm{Exp}\left\{d + c\, t - a\,\mathrm{Exp}\left\{-b\, t\right\}\right\} \delta t \tag{8}$$

Using Equations (1) and (2) above the solution to Equation (8) above is... [1]

$$P_{m,n} = \theta\, R_0 \,\mathrm{Exp}\left\{d\right\} a^{\frac{c}{b}} b^{-1}\left[\Gamma\left(-\frac{c}{b},\, a\,\mathrm{Exp}\left\{-b\, n\right\}\right) - \Gamma\left(-\frac{c}{b},\, a\,\mathrm{Exp}\left\{-b\, m\}\right)\right] \tag{9}$$

Answers To Our Hypothetical Problem

Question 1: Graph the revenue growth rate over the time interval $[0, 20]$.

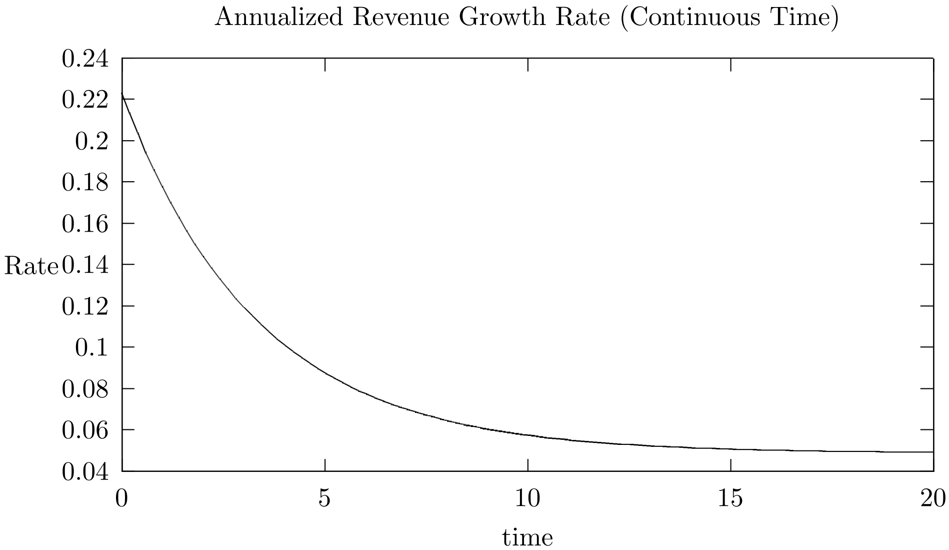
Using Equation (3) above and the parameters in Table 1 above the parameter values for our annualized revenue growth rate equation are...

$$\omega = \ln(1 + 0.05) = 0.0488 \text{ ...and... } \Delta = \ln(1 + 0.25) - \ln(1 + 0.05) = 0.1744 \text{ ...and... } \lambda = 0.30 \tag{10}$$

Using the parameter values in Equation (10) above we can write the equation for the annualized revenue growth rate (Equation (3) above) as...

$$\mu_t = 0.0488 + 0.1744 \times \mathrm{Exp}\left\{-0.30 \times t\right\} \tag{11}$$

The graph of the annualized revenue growth rate, which is the answer to question one, in continuous time is...



Question 2: What is the present value of net income realized in year 3?

Using Table 1 above the discount rate in continuous time is...

$$\kappa = \ln(1 + 0.15) = 0.1398 \tag{12}$$

Using the solution to the base equation (Equation (2) above) the integral parameter values are...

$$a = \frac{0.1744}{0.3000} = 0.5812 \ , \ b = 0.3000 \ , \ c = 0.0488 - 0.1398 = -0.0910 \ , \ d = \frac{0.1744}{0.3000} = 0.5812 \tag{13}$$

Using Equations (2) and (13) above and the parameters in Table 1 above....

$$\text{Exp}\left\{d\right\} = \text{Exp}\left\{0.5812\right\} = 1.7882 \ \text{...and...} \ a^{\frac{c}{b}} b^{-1} = 0.5812^{-\frac{0.0910}{0.3000}} \times 0.3000^{-1} = 3.9298 \tag{14}$$

Using Equations (2) and (13) above and the parameters in Table 1 above the solution to the upper incomplete gamma function where $n = 3$ is...

$$\Gamma\left(-\frac{-0.0910}{0.3000}, 0.5812 \times \text{Exp}\left\{-0.30 \times 3\right\}\right) = \Gamma\left(0.3032, 0.2363\right) = 0.9384 \tag{15}$$

Using Equations (2) and (13) above and the parameters in Table 1 above the solution to the upper incomplete gamma function where $m = 2$ is...

$$\Gamma\left(-\frac{-0.0910}{0.3000}, 0.5812 \times \text{Exp}\left\{-0.30 \times 2\right\}\right) = \Gamma\left(0.3032, 0.3190\right) = 0.7846 \tag{16}$$

Using Equations (9), (13), (14), (15) and (16) above the area under the curve, and the answer to our hypothetical problem, is...

$$P_{2,3} = 0.20 \times 1,000,000 \times 1.7882 \times 3.9298 \times \left(0.9384 - 0.7846\right) = 216,158 \tag{17}$$

References

[1] Gary Schurman, *The Incomplete Gamma Function - Part II,* December, 2017